6.3 Time Responses of Second-Order Systems:

To get the response (output) of the second-order system in the time domain, we must follow these four steps:

1) Write the standard closed-loop transfer function of the second-order system:

$$\frac{C(s)}{R(s)} = \frac{\omega_n{}^2}{s^2 + 2\zeta\omega_n s + \omega_n{}^2}$$

2) Take the Laplace transform of the input signal **r(t)** in order to get **R(s)**.
3) Substitute **R(s)** value in the above equation in order to get **C(s)**.
4) Do partial fraction of **C(s)** if required, then apply inverse Laplace transform to **C(s)** in order to get **c(t)**.

The **error signal E(s)** of this system is the difference between the input and the output. So,

$$\boldsymbol{E(s) = R(s) - C(s)}$$

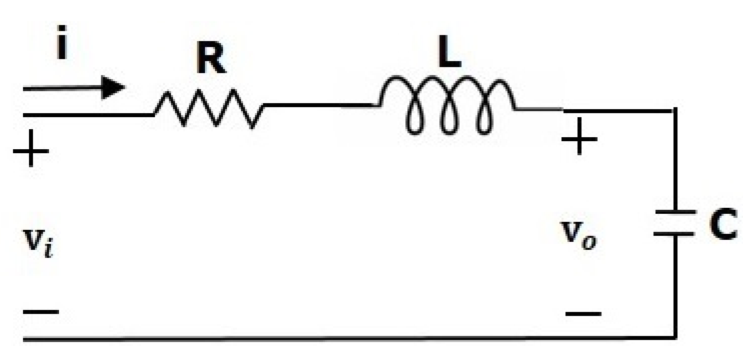
Where, **E(s)** is the Laplace transform of the error signal **e(t)**.

And after applying inverse Laplace transform on both the sides:

$$\boldsymbol{e(t) = r(t) - c(t)}$$

Example 1: A second-order series RLC circuit, shown in Figure below, has some damping due to the resistor. Determine:

1. **The key parameters for this circuit in terms of R, L, and C.**
2. **The time response and the error signal for this system if it is called under damped system.**



Solution:

1. Write the differential equation of the second-order series RLC circuit with the help of Kirchhoff's voltage law:

$$v_i = V_R + V_L + V_C$$

$$v_i = R\, i + L\frac{di}{dt} + v_o$$

And

$$v_{o(t)} = \frac{1}{C}\int i\, dt$$

Take Laplace transform for both sides of the differential equation, we get:

$$V_{i(s)} = R\, I(s) + L\, s\, I(s) + V_o(s)$$

Eliminate all but the desired variable, we get:

and $V_o(s) = \frac{1}{sC} I(s)$

or $I(s) = s\, C\, V_o(s)$

substitute I(s) in $V_{i(s)}$ equation, yields:

$$V_{i(s)} = R\,[\, s\, C\, V_o(s)] + L\, s\,[\, s\, C\, V_o(s)] + V_o(s)$$

$$V_{i(s)} = R\, C\, s\, V_o(s) + L\, C\, s^2\, V_o(s) + V_o(s)$$

$$V_{i(s)} = V_o(s)(L\, C\, s^2 + R\, C\, s + 1)$$

$$\frac{V_o(s)}{V_i(s)} = \frac{1}{L\, C\, s^2 + R\, C\, s + 1}$$

Divide the numerator and denominator by (L C):

$$= \frac{\frac{1}{L\, C}}{\frac{L\, C}{L\, C} s^2 + \frac{R\, C}{L\, C}\, s + \frac{1}{L\, C}}$$

$$\therefore \frac{V_o(s)}{V_i(s)} = \frac{\frac{1}{L\, C}}{s^2 + \frac{R}{L}\, s + \frac{1}{L\, C}}$$

Compare above equation with standard second-order equation:

$$\frac{C(s)}{R(s)} = \frac{{\omega_n}^2}{s^2 + 2\zeta\omega_n s + {\omega_n}^2}$$

$${\omega_n}^2 = \frac{1}{L\,C} \Longrightarrow \omega_n = \frac{1}{\sqrt{L\,C}}$$

$$2\zeta\omega_n = \frac{R}{L} \Longrightarrow \zeta = \frac{R}{2\,L\,\omega_n} = \frac{R}{2\,L\,\frac{1}{\sqrt{L\,C}}}$$

$$\therefore \zeta = \frac{R}{2\sqrt{\frac{L}{C}}}$$

The system is called under damped if $0 < \zeta < 1$:

$0 < \zeta < 1$

$$0 < \frac{R}{2\sqrt{\frac{L}{C}}} < 1 \Longrightarrow 0 < R < 2\sqrt{\frac{L}{C}}$$

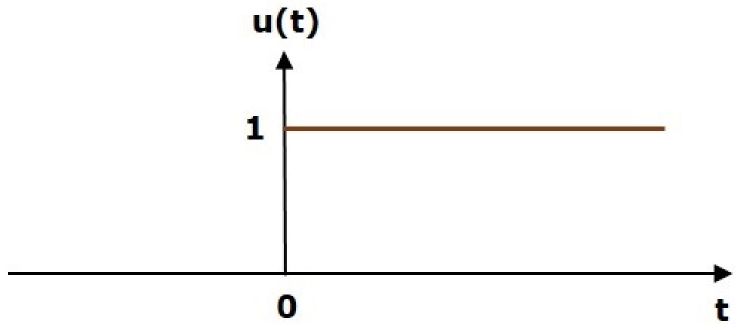
2. To get the response (output) of the second-order system in the time domain, we must follow these four steps:

1) Write the standard closed-loop transfer function of the second-order system:

$$\frac{C(s)}{R(s)} = \frac{{\omega_n}^2}{s^2 + 2\zeta\omega_n s + {\omega_n}^2}$$

2) Take the Laplace transform of the input signal **r(t)** in order to get **R(s)**.

A unit step signal **u(t)** is defined as: $\boldsymbol{r(t) = u(t)} = \begin{cases} \mathbf{1}, & \boldsymbol{t \ge 0} \\ \mathbf{0}, & \boldsymbol{t < 0} \end{cases}$



After applying Laplace transform on both the sides: $\boldsymbol{R}(\boldsymbol{s}) = \frac{\mathbf{1}}{\boldsymbol{s}}$.

3) Substitute **R(s)** value in the above equation in order to get **C(s)**.

$$C(s) = \frac{\omega_n{}^2}{s^2 + 2\zeta\omega_n s + \omega_n{}^2} R(s) = \frac{\omega_n{}^2}{s\,(s^2 + 2\zeta\omega_n s + \omega_n{}^2)}$$

4) Do partial fraction of **C(s)** if required, then apply inverse Laplace transform to **C(s)** in order to get **c(t)**.

We can modify the denominator term of the transfer function as follows:

$$s^2 + 2\zeta\omega_n s + \omega_n{}^2 = [s^2 + 2\zeta\omega_n s + \omega_n{}^2] + (\zeta\omega_n)^2 - (\zeta\omega_n)^2$$

$$= [s^2 + 2(\zeta\omega_n)s + (\zeta\omega_n)^2] + \omega_n{}^2 - (\zeta\omega_n)^2 = (s + \zeta\omega_n)^2 + \omega_n{}^2(1 - \zeta^2)$$

The transfer function becomes:

$$C(s) = \left(\frac{\omega_n{}^2}{(s + \zeta\omega_n)^2 + \omega_n{}^2(1 - \zeta^2)}\right)\left(\frac{1}{s}\right) = \frac{\omega_n{}^2}{s((s + \zeta\omega_n)^2 + \omega_n{}^2(1 - \zeta^2))}$$

Do partial fractions of C(s):

$$C(s) = \frac{\omega_n{}^2}{s((s + \zeta\omega_n)^2 + \omega_n{}^2(1 - \zeta^2))} = \frac{A}{s} + \frac{Bs + C}{(s + \zeta\omega_n)^2 + \omega_n{}^2(1 - \zeta^2)}$$

$$= \frac{A(s^2 + 2\zeta\omega_n s + \omega_n{}^2) + s\,(Bs + C)}{s((s + \zeta\omega_n)^2 + \omega_n{}^2(1 - \zeta^2))}$$

On both the sides, the denominator term is the same. So, they will get cancelled by each other. Hence, equate the numerator terms.

$$\omega_n{}^2 = A(s^2 + 2\zeta\omega_n s + \omega_n{}^2) + s\,(Bs + C)$$

$$\omega_n{}^2 = (A\,s^2 + 2A\,\zeta\omega_n s + \omega_n{}^2\,A) + Bs^2 + C\,s$$

$$\omega_n{}^2 = A\,s^2 + Bs^2 + C\,s + 2A\,\zeta\omega_n s + \omega_n{}^2\,A$$

equate the constant terms on both the sides, we get:

$$\omega_n{}^2 = \omega_n{}^2\,A\;(Divide\ both\ sides\ by\ \omega_n{}^2) \Longrightarrow \boldsymbol{A = 1}$$

substitute A = 1 and equate the coefficient of the **s** terms on both the sides, we get:

$0 = C + 2A\,\zeta\omega_n = C\ + 2(1)\,\zeta\omega_n \Longrightarrow \boldsymbol{C = -2\,\zeta\omega_n}$

substitute A = 1 and equate the coefficient of the s^2 terms on both the sides, we get:

$$0 = A + B \Longrightarrow \boldsymbol{B = -1}$$

substitute the values of A, B, and C as 1, -1, and -2$\zeta\omega_n$, respectively in partial fraction expansion of C(s), we get:

$$C(s) = \frac{1}{s} - \frac{s + 2\zeta\omega_n}{(s + \zeta\omega_n)^2 + {\omega_n}^2(1 - \zeta^2)}$$

But $s + 2\zeta\omega_n = s + \zeta\omega_n + \zeta\omega_n$

$$\therefore C(s) = \frac{1}{s} - \frac{s + \zeta\omega_n}{(s + \zeta\omega_n)^2 + {\omega_n}^2(1 - \zeta^2)} - \frac{\zeta\omega_n}{(s + \zeta\omega_n)^2 + {\omega_n}^2(1 - \zeta^2)}$$

$$C(s) = \frac{1}{s} - \frac{(s + \zeta\omega_n)}{(s + \zeta\omega_n)^2 + \left(\omega_n\sqrt{1 - \zeta^2}\right)^2} - \frac{\zeta}{\sqrt{1 - \zeta^2}}\left(\frac{\omega_n\sqrt{1 - \zeta^2}}{(s + \zeta\omega_n)^2 + \left(\omega_n\sqrt{1 - \zeta^2}\right)^2}\right)$$

substitute the damped natural frequency $\boldsymbol{\omega_d = \omega_n\sqrt{\zeta^2 - 1}}$ in equation above, we get:

$$C(s) = \frac{1}{s} - \frac{(s + \zeta\omega_n)}{(s + \zeta\omega_n)^2 + {\omega_d}^2} - \frac{\zeta}{\sqrt{1 - \zeta^2}}\left(\frac{\omega_d}{(s + \zeta\omega_n)^2 + {\omega_d}^2}\right)$$

Apply inverse Laplace transform on both the sides, we get:

$$c(t) = \left(1 - e^{-\zeta\omega_n t}\cos(\omega_d t) - \frac{\zeta}{\sqrt{1 - \zeta^2}}e^{-\zeta\omega_n t}\sin(\omega_d t)\right) u(t)$$

$$c(t) = \left(1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1 - \zeta^2}}\left(\sqrt{1 - \zeta^2}\cos(\omega_d t) + \zeta\sin(\omega_d t)\right)\right) u(t)$$

If $\sqrt{1 - \zeta^2} = \sin(\theta)$, then ζ will be $\cos(\theta)$. Substitute these values in the above equation, we get:

$$c(t) = \left(1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1 - \zeta^2}}(\sin(\theta)\cos(\omega_d t) + cos(\theta)\sin(\omega_d t))\right) u(t)$$

$$\therefore c(t) = \left(1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1 - \zeta^2}}\sin(\omega_d t + \theta)\right) u(t)$$

So, the unit step response of the second order system is having damped oscillations when ζ lies between zero and one.

The error signal $e(t)$ for this system is:

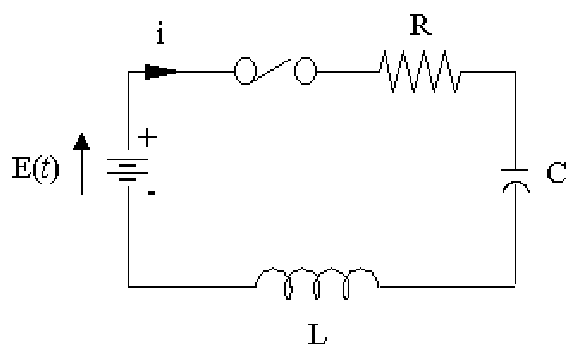
$$\boldsymbol{e(t) = r(t) - c(t)}$$

$$\therefore e(t) = u(t) - \left(1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin(\omega_d t + \theta)\right) u(t)$$

$$e(t) = \left(\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin(\omega_d t + \theta)\right) u(t)$$

Note: It is desirable that the transient response be sufficiently fast and be sufficiently damped. Thus, for a desirable transient response of a second-order system, the damping ratio must be between 0.4 and 0.8. Small values of ζ ($\zeta < 0.4$) yield excessive overshoot in the transient response, and a system with a large value of ζ ($\zeta > 0.8$) responds sluggishly.

Example 2: Consider a second-order series RLC system shown in Figure below. Determine the type of this system if R=10 Ω, L=1 H, and C=62.5 mF?



Solution:

Write the differential equations with the help of Kirchhoff's voltage law:

$$E = V_L + V_R + V_C$$

$$E = L\frac{d\,i}{dt} + R\,i + \frac{1}{C}\int i\,dt$$

Differentiating, we get:

$$0 = L\frac{d^2\, i}{dt^2} + R\,\frac{d\,i}{dt} + \frac{1}{C}\,i$$

Taking Laplace transform, we get:

$$0 = L\,s^2 I(s) + R\,s\,I(s) + \frac{1}{C}\,I(s)$$

Divide both sides of the equation by (L I(s)), we get:

$$0 = s^2 + \frac{R}{L}\,s + \frac{1}{L\,C}$$

But R=10 Ω, L=1 H, and C=62.5 mF

$$0 = s^2 + \frac{10\,\Omega}{1\text{ H}}\,s + \frac{1}{1\text{ H} \times 62.5\text{ mF}}$$

$$0 = s^2 + 10\,s + 16$$

Compare above equation with the characteristic equation $s^2 + 2\zeta\omega_n s + {\omega_n}^2 = 0$, yields:

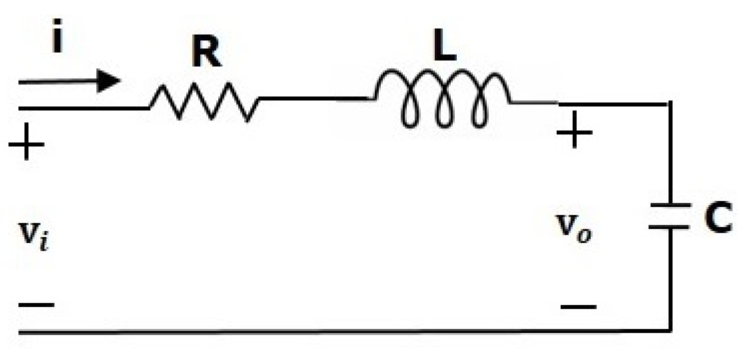
$${\omega_n}^2 = 16 \implies \omega_n = 4\,rad/sec$$

$$2\zeta\omega_n = 10 \implies \zeta = \frac{10}{2\ \omega_n} = \frac{10}{2\ \times 4} = 1.25$$

$\because \zeta > 1$, the system is called over damped.

Homework1: A second-order series RLC circuit, shown in Figure below, has some damping due to the resistor. Determine:

1. **The key parameters for this circuit in terms of R, L, and C.**
2. **The time response and the error signal for this system if it is called undamped system.**



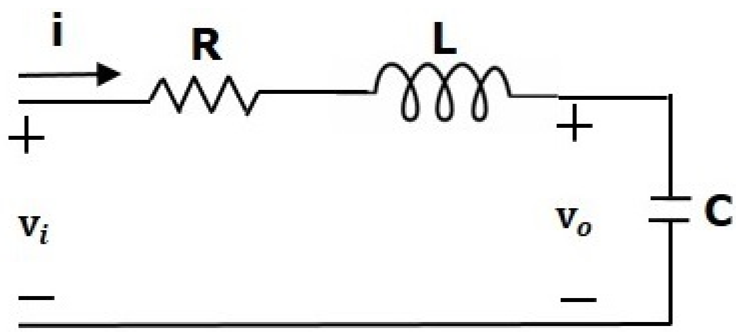
Answer:

$$c(t) = (1 - cos\omega_n t)\, u(t)$$

$$e(t) = cos\omega_n t\, u(t)$$

Homework2: **A second-order series RLC circuit, shown in Figure below, has some damping due to the resistor. Determine:**

3. **The key parameters for this circuit in terms of R, L, and C.**
4. **The time response and the error signal for this system if it is called critically damped system.**



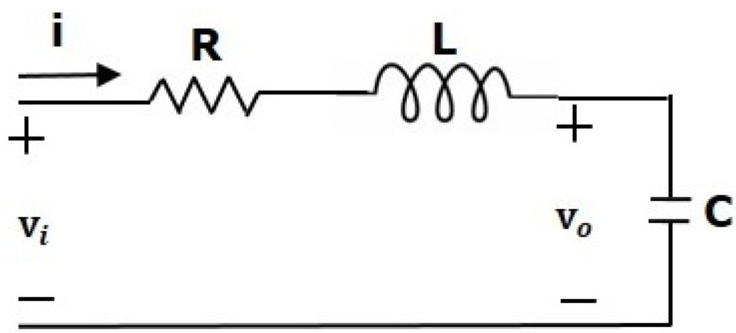
Answer:

$$c(t) = (1 - e^{-\omega_n t} - \omega_n t e^{-\omega_n t})\, u(t)$$

$$e(t) = (e^{-\omega_n t} + \omega_n t e^{-\omega_n t})\, u(t)$$

Homework3: **A second-order series RLC circuit, shown in Figure below, has some damping due to the resistor. Determine:**

1. **The key parameters for this circuit in terms of R, L, and C.**
2. **The time response and the error signal for this system if it is called over damped system.**



Hint:

We can modify the denominator term of the transfer function as follows:

$$s^2 + 2\zeta\omega_n s + {\omega_n}^2 = [s^2 + 2\zeta\omega_n s + {\omega_n}^2] + (\zeta\omega_n)^2 - (\zeta\omega_n)^2$$

$$= [s^2 + 2(s)(\zeta\omega_n) + (\zeta\omega_n)^2] + {\omega_n}^2 - (\zeta\omega_n)^2 = (s + \zeta\omega_n)^2 - {\omega_n}^2(\zeta^2 - 1)$$

Answer:

$$c(t) = \begin{pmatrix} 1 + \left(\dfrac{1}{2\left(\zeta + \sqrt{\zeta^2 - 1}\right)\left(\sqrt{\zeta^2 - 1}\right)}\right) e^{-\left(\zeta\omega_n + \omega_n\sqrt{\zeta^2-1}\right)t} \\ -\left(\dfrac{1}{2(\zeta - \sqrt{\zeta^2 - 1})(\sqrt{\zeta^2 - 1})}\right) e^{-(\zeta\omega_n - \sqrt{\zeta^2-1})t} \end{pmatrix} u(t)$$

$$e(t) = \begin{pmatrix} -\left(\dfrac{1}{2\left(\zeta + \sqrt{\zeta^2 - 1}\right)\left(\sqrt{\zeta^2 - 1}\right)}\right) e^{-\left(\zeta\omega_n + \omega_n\sqrt{\zeta^2-1}\right)t} \\ +\left(\dfrac{1}{2(\zeta - \sqrt{\zeta^2 - 1})(\sqrt{\zeta^2 - 1})}\right) e^{-(\zeta\omega_n - \sqrt{\zeta^2-1})t} \end{pmatrix} u(t)$$